

# MINIMAL TYPES IN STABLE BANACH SPACES

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**ABSTRACT.** We prove existence of wide types in a continuous theory expanding a Banach space, and density of minimal wide types among stable types in such a theory. We show that every minimal wide stable type is “generically” isometric to an  $\ell_2$  space. We conclude with a proof of the following formulation of Henson’s Conjecture: every model of an uncountably categorical theory expanding a Banach space is prime over a spreading model, isometric to the standard basis of a Hilbert space.

## 1. INTRODUCTION

The main motivation for this work is a conjecture formulated by C. Ward Henson in the 1970s concerning geometric structure of non-separably categorical elementary classes of Banach spaces. Recently, after some partial progress had been made on Henson’s question, the second author suggested a formulation of the conjecture, which seemed within reach. In this paper, we prove a more general result. Our techniques suggest the beginning of geometric structure theory for a larger class of *stable* elementary classes of Banach spaces.

We remind the reader that a class  $K$  of Banach spaces is called *elementary* if it is closed “nicely” under the ultra-product construction. More precisely,  $K$  is elementary if it is closed under isometries, ultra-products and ultra-roots. It is widely understood that analyzing ultra-products and ultra-powers of a Banach space can be very helpful (and often essential) for understanding its local structure. This suggests that it is natural to consider a Banach space *together with all its ultra-powers* – that is, even if one is only interested in a specific space, it can be instructive to look at the elementary class that it generates. Hence elementary classes of Banach spaces are interesting objects.

Equivalently, a class of Banach spaces is elementary if it can be axiomatized in an appropriate logic. One can work with either Henson’s logic of positive bounded formulae [Hen76] or continuous first order logic [BYU10].

Many “natural” classes of Banach spaces are elementary, for example:

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- Fix  $1 \leq p < \infty$ . Then the class of all Banach spaces isometric to  $L_p(\mu)$  for some measure  $\mu$  is elementary.
- The class of all Banach spaces whose dual is isometric to  $L_1(\mu)$  for some measure  $\mu$  is elementary.
- The class of all Banach spaces isometric to  $C(K)$  for some compact Hausdorff space  $K$  is elementary. In this case the precise axiomatization is not known (but it has been shown that this class is closed under ultra-products and ultra-roots).

An elementary class of Banach spaces  $K$  is called *categorical* in a cardinal  $\lambda$  if there is a unique  $B \in K$  of density character  $\lambda$  up to isometry. A class  $K$  is called *uncountably categorical*, or *non-separably categorical*, if it is categorical in some uncountable  $\lambda$ . The most basic example is the class of all Banach spaces isometric to a Hilbert space. There are other known examples, but in all of them the behavior of the class is “controlled” in a very strong sense by an underlined Hilbert space. It is also surprisingly hard to find such examples, as well as to prove categoricity.

The following has been shown recently by Henson and Raynaud:

*Example 1.1.* Let  $E$  any finite-dimensional Banach space, and let  $H(E)$  be the class of all Banach spaces which are isometric to the direct sum of an infinite dimensional Hilbert space with  $E$ . Then  $H(E)$  is elementary and categorical in all infinite density characters.

C. Ward Henson conjectured the following

**Conjecture 1.2.** (Henson) Let  $K$  be an uncountably categorical elementary class of Banach spaces. Then

- $K$  is categorical in all uncountable cardinalities.
- Any  $B \in K$  of uncountable density character is “very close” to (and “determined by”) an underlined Hilbert space.

The first part of the Conjecture is simply an analogue of a well-known Łoś’s Conjecture in classical logic. It was established independently by the authors [SU11], and by Itai Ben Yaacov [BY05]. Both proofs resembled classical proofs of analogous results in the first order context. In this paper we prove a version of the more interesting (the second) part of Henson’s Conjecture. Our main theorem is

**Theorem 1.3.** *Let  $K$  be an uncountably categorical elementary class of Banach spaces. Then there is a separable  $B_0 \in K$  and a minimal wide type  $p_0$  over  $B_0$ , such that*

- *Any Morley sequence in  $p_0$  is isometric to the standard basis of a Hilbert space.*
- *Any non-separable  $B \in K$  is prime over a Morley sequence in  $p_0$  (which is a basis of a spreading model of  $B_0$ ).*

We explain all the necessary terms that appear in the statement later.

In fact, our results are somewhat more general: we prove the theorem above for any elementary class of Banach spaces expanded with continuous extra-structure (we explain the different contexts we work in in section 2).

The following corollary, which is more accessible to the general audience, can be stated.

**Theorem 1.4.** *Let  $K$  be an uncountably categorical elementary class of Banach spaces. Then any non-separable  $B \in K$  is prime over a sequence isometric to the standard basis of a Hilbert space (which is a spreading model of a fixed separable  $B_0$  in  $K$ ).*

It should be remarked that Theorem 1.4 is significantly easier than Theorem 1.3: an attentive reader should be able to deduce it directly from Dvoretzky-Milman Theorem (Fact 3.4) and compactness (the “spreading model” part requires techniques of basic stability, but is still straightforward). One of the main features of Theorem 1.3 is that it gives a definable geometric object (a wide type) that generates the basis for the Hilbert space that underlines any non-separable (in fact, any “large enough”) member of  $K$ . One may ask whether stronger definability requirements hold: for example, whether the Hilbert space itself may be assumed to be type-definable. This seems to be a hard and interesting question.

Let us explain some of the basic notions that appear in the statements above.

A model  $B \in K$  is called *prime* over a set  $A$  if whenever  $A$  embeds into  $B' \in K$  via  $f: A \hookrightarrow B'$ , there is an embedding of  $B$  into  $B'$  that extends  $f$ . A model  $B_0 \in K$  is called *prime* if it is prime over the empty set. Note that all the embedding in this case are *isometries*.

We remind the reader that a sequence  $\langle e_i : i < \lambda \rangle$  is called a *spreading model* of a Banach space  $B$  [BS74] if it is 1-subsymmetric (quantifier-free indiscernible), and there is a sequence  $\langle b_n : n < \omega \rangle$  in  $B$  which is asymptotically isometric to  $\langle e_i : i < \lambda \rangle$ : there exists a null sequence of positive reals  $\langle \varepsilon_\ell : \ell < \omega \rangle$  such that whenever  $k > \ell$ , we have

$$\left| \left\| \sum_{j=0}^{k-1} r_j b_{n_j} \right\| - \left\| \sum_{j=0}^{k-1} r_j e_j \right\| \right| < \varepsilon_\ell$$

for every  $\ell < n_0 < n_1 < \dots < n_{k-1} < \omega$  and  $r_j \in [-1, 1]$ .

Clearly, since  $\langle e_n \rangle$  is 1-subsymmetric, the sum  $\sum_{j=0}^{k-1} r_j e_j$  can be replaced with  $\sum_{j=0}^{k-1} r_j e_{i_j}$  for every  $i_0 < i_1 < \dots < i_{k-1} < \lambda$ .

Equivalently, in model theoretic terms, a sequence  $\langle e_i : i < \lambda \rangle$  is called a spreading model of  $B$  if it is a quantifier free “co-heir” sequence over  $B$ : that is, it is quantifier free indiscernible, and  $\text{tp}(e_i / B e_{<i})$  is finitely satisfiable in  $B$  (here “tp” stand for the quantifier free type in the pure language of normed spaces).

So Theorem 1.3 shows that every non-separable  $B \in K$  is essentially prime over a Hilbert space, which is nicely based on a “small” separable “base” space. In Example 1.1 the base space is the direct sum of  $E$  with a separable Hilbert space.

Theorem 1.3 is proven in section 5, Theorem 5.4.

Although it was not clear to us until the proofs were basically finalized, a posteriori it has become transparent that this formulation of Henson’s Conjecture, and the techniques developed on the way to its proof, provide in a sense a true analogue of geometric

characterizations of uncountably categorical elementary classes in classical model theory, continuing the work of Baldwin, Lachlan, Zilber and others (which we discuss in the next subsection) in the context of Banach spaces. We believe that this paper lays the foundations for the developing of geometric stability in this setting. Hence it is our hope that the results here are not of isolated interest, but rather a beginning of a new chapter in model theoretic study of Banach spaces.

**History and background.** Henson’s Conjecture is strongly related to well-known results on classical uncountably categorical elementary classes. In 1962 Morley [Mor65] proved the conjecture of Loś which stated that a countable first order theory  $T$  which is categorical in some uncountable power, is categorical in any uncountable power. Basic examples of such theories are the theory of algebraically closed fields of a fixed characteristic, and the theory of vector spaces over a fixed field. Morley’s proof showed that an uncountably categorical theory  $T$  admits a *notion of independence* and that any model of  $T$  is both saturated (“rich”) and prime (“small”) over a basis with respect to this notion.

Less than ten years later Baldwin and Lachlan [BL71] gave a different, more geometric proof of Morley’s Theorem. They showed that every model of an uncountably categorical theory  $T$  is determined by a “strongly minimal” definable set, on which the independence notion is of a very special kind: it is determined by algebraic closure. Their proof also gave information about countable models of uncountably categorical theories.

The results of Baldwin and Lachlan led to further research. Specifically, Zilber studied geometric structure of strongly minimal sets and showed that in many cases they are either “field-like” or “group-like” (and in the “field-like” case one can interpret an algebraically closed field in the model). One reference for Zilber’s work is [Zil93]. A posteriori it turns out that Henson’s Conjecture called for a similar analysis for Banach spaces (“interpreting” a Hilbert space inside the model), but no appropriate tools were available until very recently. For example, no analogue of a strongly minimal set was known. In this article, we introduce new geometric objects, which we call *wide types*. Our thesis is that *minimal wide types* are the correct analogue of strongly minimal sets in this setting.

Another important notion that we are going to make use of is *stability*. In his proof, Morley introduced the notion of  $\omega$ -*stability*. He proved that an uncountably categorical theory is  $\omega$ -stable, and that  $\omega$ -stability implies several good properties, such as existence of prime models over any set (we shall explain the notion of a prime model later) and a “nice” notion of independence. Later Shelah defined the more general notion of *stability* and showed that any stable theory admits a similar notion of independence.

Stability was first introduced to functional analysis by Krivine and Maurey, who proved in [KM81] that any stable Banach space contains an almost isometric copy of  $\ell_p$  for some  $p$ . It was further investigated by Iovino in [Iov99a, Iov99b] and other works, and, more recently, by Ben Yaacov and the authors (e.g. [BY05, BYU10, SU11]).

We have already pointed out that the first part of Henson’s Conjecture states that the analogue of Morley’s Theorem holds for classes of Banach spaces. This was proven

independently by the authors [SU11] and Itai Ben Yaacov [BY05]. The two proofs are quite different, but none of them gives much geometric information. In some sense, both correspond to Morley’s original proof, and do not provide “Baldwin-Lachlan analysis”. We will use several results from [SU11] in this article. In particular, we will use the fact that uncountable categoricity implies a topological version of  $\omega$ -stability, which has property similar to those of classical  $\omega$ -stability. Consequently, uncountably categorical classes of Banach spaces are stable.

We would also like to mention the classical theorem of Macintyre [Mac71]: any  $\omega$ -stable field is algebraically closed. In a sense, this is a “dual” result to Morley Theorem: it shows that “algebraic” structure follows from model-theoretic properties. The second part of Henson’s Conjecture has a similar flavor.

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## 2. PRELIMINARIES

In this section we describe the framework in which we are going to work. A reader who is familiar with continuous logic can easily skip to the last subsection (subsection 2.3).

We refer the reader to [Hei80], [HI02] or [BYBHU08] for the definition of an ultraproduct of Banach spaces, and, more general, normed structures.

In subsection 2.1 we describe the very basic framework of quantifier free formulas in the pure language of Banach spaces, which is enough for proving Theorem 1.3. The presentation in subsection 2.1 should be accessible to any mathematician, and very familiar to a functional analyst. For example, quantifier free types in this basic frameworks are precisely what Krivine and Maurey defined as “types” in [KM81].

In subsection 2.2 we present the more general context of continuous logic expanding the Banach space structure. Working in this framework, we prove more general results.

### 2.1. The basic case.

**Definition 2.1.** • A *quantifier free formula in the pure language of Banach spaces* over a set  $A$  is an expression of the form  $\|x + a\|$  where  $x$  is a variable and  $a \in A$ . We call such a formula a *pure q.f. formula*.

- A *pure q.f. condition* over  $A$  is an expression of the form  $\varphi(x) = r$  where  $\varphi(x)$  is a pure q.f. formula over  $A$  and  $r \in \mathbb{R}$ .
- Let  $\Sigma$  be a collection of pure q.f. conditions over a set  $A$ ,  $A \subseteq M$ ,  $M$  a Banach space. We say that  $\Sigma$  is *approximately finitely satisfiable* in  $M$  if for every finite  $\Sigma_0 \subseteq \Sigma$  and  $\varepsilon > 0$ , there is  $b \in M$  such that for every  $[\varphi(x) = r] \in \Sigma_0$  we have  $\varphi(b) \in [r - \varepsilon, r + \varepsilon]$ .

- Given a Banach space  $M$  and a subset  $A \subseteq M$ , a *pure q.f. partial type* in  $M$  over  $A$  is a collection  $\pi(x)$  of pure q.f. conditions which is approximately finitely satisfiable in  $M$ , such that in addition  $[\|x\| = r] \in \pi$  for some  $r \in \mathbb{R}$ .
- Given a partial type  $\pi(x)$ , we say that the value of the formula  $\varphi(x)$  is *determined* by  $\pi$  if  $[\varphi(x) = r] \in \pi$  for some  $r \in \mathbb{R}$ . Otherwise we say that the value of  $\varphi$  is *undetermined* by  $\pi$ .
- Given a Banach space  $M$  and a subset  $A \subseteq M$ , a *complete pure q.f. type* in  $M$  over  $A$  is a partial pure q.f. type in  $M$  over  $A$  which determines the value of any pure q.f. free formula over  $A$ .  
In other words, a complete pure q.f. type  $p$  over  $A$  can be (and often is) viewed as a function  $\tau: A \rightarrow \mathbb{R}$  such that for any  $a \in A$  we have  $\tau(a) = r$  if and only if  $[\|x + a\| = r] \in p$ .
- We denote the space of all complete pure q.f. types in  $M$  over  $A$  by  $S_{qf}(A, M)$ , or just  $S_{qf}(A)$  when  $M$  is clear from the context.
- Given a partial type  $\pi(x)$  and a formula  $\varphi(x)$ , we denote by  $\varphi^\pi$  the *value of  $\varphi$  according to  $\pi$* . In other words,  $\varphi^\pi = r$  iff  $[\varphi(x) = r] \in \pi(x)$ . Of course, this only makes sense if  $\pi$  determines the value of  $\varphi$ .
- We say that  $b \in M$  *realizes* a partial type  $\pi(x)$  if  $\varphi(b) = \varphi^\pi$  for every formula  $\varphi(x)$  (whose value is determined by  $\pi$ ).

The following version of Compactness Theorem can be found in e.g. [HI02].

**Fact 2.2.** *Let  $M$  be a Banach space,  $\pi(x)$  a partial type in  $M$  over  $A$ . Then there exists an ultra-power  $\hat{M}$  of  $M$  and  $b \in \hat{M}$  such that  $b$  realizes  $\pi$ .*

**Definition 2.3.** We call a Banach space *qf-saturated* if for every  $A$  of cardinality less than the density character of  $M$  and every  $p \in S_{qf}(A, M)$ ,  $p$  is realized in  $M$ .

Given an elementary class of Banach spaces, we will assume the following:

*There exists a Banach space  $\mathfrak{C}$ , which is qf-saturated, and whose cardinality is much bigger than all other cardinals discussed in this paper, and all  $M \in K$ , which are of interest to us, are subspaces of  $\mathfrak{C}$ .*

Such  $\mathfrak{C}$  is called the *monster model* of  $K$ . There are slight set-theoretic assumptions which are involved in the existence of monster models, but we will not be concerned with these issues here. In fact, in the cases that we are interested in in this paper (e.g. if  $K$  is uncountably categorical, or just stable), no such assumptions are necessary.

**2.2. The general case.** In this subsection we give a very quick overview of continuous logic in the special case of normed structures. The reader is referred to [BYU10] or [BYBHU08] for details.

Just like in classical logic, a continuous *signature* consists of constant symbols, function symbols and predicate symbols. There is a special predicate symbol for the norm,  $\|\cdot\|$ . Each function symbol and predicate symbol is equipped with its arity  $k \in \mathbb{N}$  and its modulus of uniform continuity, which is a continuous function  $\delta$  from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  with

$\delta(0) = 0$ . We will always assume that the signature contains the signature of a vector space over  $\mathbb{Q}$ ; that is, it contains a constant symbol 0, a 2-ary function for vector addition, and for every  $q \in \mathbb{Q}$ , a 1-ary function  $\cdot_q(x)$  for multiplication by  $q$ .

A continuous *pre-structure*  $M$  for a given signature is a semi-normed space, in which all the constant symbols are interpreted as elements, function symbols - as functions on the structure, predicate symbols - as functions from the structure to  $\mathbb{R}$ . More precisely, if  $P$  is a predicate symbol of arity  $k$ , then its interpretation  $P^M$  is a function  $P^M: M^k \rightarrow \mathbb{R}$ . Similarly, if  $f$  is a function symbol of arity  $k$ , then its interpretation  $f^M$  is a function  $f^M: M^k \rightarrow M$ .

Moreover, we demand that the predicate  $\|\cdot\|$  is interpreted as a semi-norm on  $M$  and all the predicates and functions are uniformly continuous with respect to  $\|\cdot\|$ , respecting their continuous moduli. This ensures that the predicates and functions are continuous uniformly over all structures. Roughly speaking, this is what is needed in order to make ultraproducts work.

A *structure* is a pre-structure in which  $\|\cdot\|$  is a complete norm.

One notion which is important to understand in order to read the paper in full generality is that of a *formula*. The algebra of formulas is obtained as follows. An *atomic* formula is an expression of the form  $P(\tau_1, \dots, \tau_k)$  where  $P$  is a predicate symbol of arity  $k$ , and every  $\tau_i$  is a *term*, which is a “generalized” function symbol (an expression that can be obtained by composing existing function symbols and applying them to variables and constants).

For example, quantifier free formulas discussed in subsection 2.1, which are expressions of the form  $\|x + y\|$ ,  $\|x + a\|$  (where  $a$  is a constant) or, more generally,  $\|\sum_{i < n} q_i x_i + a\|$  (where  $q_i \in \mathbb{Q}$ ,  $x_i$  are variables) are atomic formulas. Note that  $qx$  means  $\cdot_q(x)$ , so we omit the formal function symbol and use the familiar notation.

Now the algebra of formulas is the closure of the collection of atomic formulas under “connectives” - bounded continuous function from  $\mathbb{R}^k \rightarrow \mathbb{R}$  (for some  $k \in \mathbb{N}$ ), “quantifiers”  $\sup_x$  and  $\inf_x$  (where  $x$  is a variable) and uniform limits. Note that due to uniform limits we obtain formulas of the form  $\|rx\|$  where  $r \in \mathbb{R}$ , and due to connectives we can for example speak of a formula  $r \cdot \|x\|$  or  $|\|x\| - r|$ , where  $r \in \mathbb{R}$ . Using quantifiers, we get formulas of the form

$$\sup_x |\|x\| - r|$$

The collection of all formulas (for a given signature) is also called a *language*.

A (closed) *condition* is an expression of the form  $[\varphi \in C]$ , where  $\varphi$  is a formula and  $C$  is a closed subset of  $\mathbb{R}$ . We will only work with conditions where  $C$  is a closed interval, often a point (most of the time  $C = \{0\}$ ).

A variable in a formula  $\varphi$  is called *bounded* if it is in a scope of a quantifier, and it is called *free* if it is not bounded. Given a formula  $\varphi$  with free variables  $x_1, \dots, x_k$ , we often write  $\varphi(x_1, \dots, x_k)$  in order to emphasize the free variables. It is easy to see that a formula  $\varphi(x_1, \dots, x_k)$  and a structure  $M$ , defines a function  $\varphi^M: M^k \rightarrow M$ . In fact,  $\varphi^M$  is

uniformly continuous, and, moreover, uniformly so in all structures (one can calculate the uniform continuity modulus of  $\varphi$ , given the moduli of all function and predicate symbols in the signature). So given a formula  $\varphi(x_1, \dots, x_k)$ , a structure  $M$ , and  $a_1, \dots, a_k \in M$ , one can calculate  $\varphi^M(a_1, \dots, a_k) \in \mathbb{R}$ . Hence given a condition  $[\varphi(x_1, \dots, x_k) \in C]$ , a structure  $M$ , and  $a_1, \dots, a_k \in M$ , it makes to ask whether the condition is *true* in  $M$  (denoted by  $M \models [\varphi \in C]$ ). If  $M \models [\varphi \in C]$ , we also say that  $M$  is a *model* of (for) this condition.

A *theory* is a collection of conditions with no free variables, which has a model. We normally assume that a theory  $T$  is closed under entailment, that is, if a condition  $[\varphi \in C]$  follows from  $T$  (which means that it is true in all models of  $T$ ), then  $[\varphi \in C] \in T$ . Compactness Theorem (see [HI02, BYBHU08]) states that a collection of conditions has a model if and only if every finite subset of it does. A theory is called *complete* if for every condition  $[\varphi \in C]$  either it is in  $T$  or for some closed  $D \subset \mathbb{R}$  disjoint to  $C$  we have  $[\varphi \in D] \in T$ . Equivalently,  $T$  is complete if it “forces” a value for every formula  $\varphi$  with no free variables, that is,  $[\varphi = r] \in T$  for some  $r \in \mathbb{R}$ . We will denote that value by  $\varphi^T \in \mathbb{R}$ . Note that every theory can be extended to a complete theory (in fact, every model  $M$  of  $T$  determines a complete theory).

We will normally assume that we have a fixed complete theory in the background, and all structures are models of  $T$ ; we will therefore often simply call them “models”. Given a model  $M$ , and a subset  $A$  of  $M$ , we will often expand the language by adding constant symbols for all elements of  $A$ . Call this language  $L(A)$ . Then  $M$  naturally becomes an  $L(A)$ -structure; we will call  $L(A)$ -formulas “formulas over  $A$ ”.

The next definition is of central importance. A *type*  $\pi(x)$  in a model  $M$  over a set  $A$  is a collection of conditions of the form  $\varphi(x) \in [r_\varphi, s_\varphi]$  (where  $\varphi$  is a formula over  $A$ ,  $r_\varphi, s_\varphi \in \mathbb{R}$ ), which is finitely approximately satisfiable in  $M$ . The latter means that for every finite subset  $\pi_0(x)$  of  $\pi(x)$  and for every  $\varepsilon > 0$  there exists  $a \in M$  such that  $\varphi^M(a) \in [r_\varphi - \varepsilon, s_\varphi + \varepsilon]$  for every condition  $\varphi(x) \in [r_\varphi, s_\varphi]$  in  $\pi_0(x)$ . Equivalently, by Compactness, a type  $\pi(x)$  is a collection of conditions of the form above such that there is an ultrapower  $\hat{M}$  of  $M$  and  $a \in \hat{M}$  which satisfies all the conditions in  $\pi(x)$ . We say that  $a$  realizes  $\pi$  and write  $a \models \pi$ .

Note that in general  $x$  in the definition of the type does not need to be a singleton (so neither does  $a$ , that is, maybe  $a \in M^k$  for some  $k \in \mathbb{N}$ ), although in this paper it we will normally work with formulas and types in one variable.

We say that a type  $\pi(x)$  *determines* a value of a formula  $\varphi(x)$  if  $[\varphi(x) = r] \in \pi$  for some  $r \in \mathbb{R}$ . A *complete* type over  $A$  is a type over  $A$  which determines a value for every formula over  $A$  (with the right number of variables). We will denote the value of a formula  $\varphi$  “according to the type  $\pi$ ” by  $\varphi^\pi$ .

Note the correspondence between complete theories and complete types (a complete type can be viewed as a complete theory in a certain language). We denote the space of all complete type over a set  $A$  in  $n$  variables by  $S^n(A)$ . This is a compact Hausdorff topological space, but we will not be concerned with this fact here. Let  $S(A) = \bigcup_{n < \omega} S^n(A)$ .



Note that the space of types is relative to a certain model which contains  $A$ ; but as we'll see in a bit, we will be working in one big model of the theory  $T$  (the “monster” model), and all types will be computed in that structure.

Given a model  $M$ , a set  $A$  and a tuple  $a \in M^k$ , we denote by  $\text{tp}(a/A)$  the collection of all closed conditions over  $A$  that  $a$  satisfies. It is easy to see that  $\text{tp}(a/A) \in S(A)$ , and we call it the type of  $a$  over  $A$  (again, we forget to mention  $M$ ). Conversely, every complete type over a set  $A$  is the type of some  $a$  over  $A$  (possibly  $a$  is in some ultrapower of  $M$ ; soon this won't matter because in the “monster” model we will have realizations for all types over “small” sets).

Given a cardinal  $\lambda$ , a model  $M$  is called  $\lambda$ -saturated if every type over a subset of  $M$  of cardinality less than  $\lambda$  is realized in  $M$ . A model is called *saturated* if it is  $|M|$ -saturated. There is a mild set-theoretic assumption that goes into the existence of saturated model, and it can be avoided if one works with a slightly weaker notion than saturation (which has the same properties that we care about), but we will not go into the details here. As a matter of fact, in the cases that we will be interested in in this paper (e.g.  $T$  uncountably categorical, or just stable), saturated model provably exist. Given a (complete) theory  $T$ , we will assume the following:

*There exists a saturated model  $\mathfrak{C}$  of cardinality  $\kappa^*$  for some big enough cardinal  $\kappa^*$ , that is, much bigger than all cardinals mentioned in this paper (except  $\kappa^*$  itself, of course). We call  $\mathfrak{C}$  the “monster model” of  $T$ .*

A useful consequence of saturation is the following homogeneity property of the monster model: given two tuples  $a, b \in M^k$  and a set  $A$  (of “small” cardinality, that is, less than  $\kappa^*$ ),  $\text{tp}(a/A) = \text{tp}(b/B)$  if and only if there exists  $\sigma \in \text{Aut}(\mathfrak{C}/A)$  (the group of automorphisms of  $\mathfrak{C}$  fixing  $A$  pointwise) such that  $\sigma(a) = b$ .

Another useful notion (although we will not really need it here) is that of an *elementary submodel*: if  $M$  is a substructure of  $N$ , we say that  $M$  is *elementary* in  $N$ ,  $M \prec N$ , if for any formula  $\varphi$  over  $M$  with no free variables, we have  $\varphi^M = \varphi^N$ . For example,  $M$  is always elementary in any of its ultrapowers (this is Łoś's Theorem adapted to this context; see [BYU10, BYBHU08]).

The monster model of  $T$  embeds elementarily any  $M \models T$  of “small” cardinality. This is why we will be able to assume that all models of  $T$  are elementary submodels of  $\mathfrak{C}$ . Moreover, types over subsets of  $M$  are the same in  $M$  and any elementary extension; so it will be enough to talk about types in  $\mathfrak{C}$  (and we will not mention it).

**2.3. Context.** The general context:  $T$  is a continuous theory, whose monster  $\mathfrak{C}$  expands a Banach space  $\mathfrak{B}$ . We denote the language of  $T$  by  $L = L_{\mathfrak{C}}$  and the language of Banach spaces (which is a part of  $L$ ) by  $L_{\mathfrak{B}}$ .

As we mentioned before, one can restrict themselves to the following context:  $K$  is an elementary class of Banach spaces,  $\mathfrak{C} = \mathfrak{B}$  is its monster model.

As usual, all sets and tuples mentioned in the paper are subsets of  $\mathfrak{C}$  (of cardinality less than  $|\mathfrak{C}|$ ), and all models are elementary submodels of  $\mathfrak{C}$  (again, of “small” cardinality).

## 3. WIDE TYPES OVER BANACH SPACES

Recall that  $\mathfrak{C}$  expands a real Banach space  $\mathfrak{B}$ .

**Definition 3.1.** We call a partial type in 1 variable  $\pi(x)$  (possibly with parameters) *wide* if the set of realizations of  $\pi(x)$  in  $\mathfrak{C}$  contains the unit sphere of an infinite dimensional subspace of  $\mathfrak{B}$ .

*Remark 3.2.* The type  $x = x$  is wide.

The main goal of this section is showing that complete wide types exist over any set. We will make use of the following well-known result, which is sometimes referred to as Concentration of Measure Phenomenon, or the Dvoretzky-Milman-Ramsey Phenomenon. It is a consequence of the renown Dvoretzky's Theorem [Dvo61], but the approach we take is due to Milman, e.g. [Mil92], Theorem 1.2. We will refer to this fact as Dvoretzky-Milman Theorem.

**Definition 3.3.** (i) Let  $B$  be a Banach space,  $\mathbf{S}(B)$  the unit sphere of  $B$ ,  $f: \mathbf{S}(B) \rightarrow \mathbb{R}$ . The *spectrum*  $\gamma(f)$  is the collection of all  $r \in \mathbb{R}$  such that for every  $\varepsilon > 0$  and any integer  $k$  there exists a  $k$ -dimensional subspace  $F$  of  $B$  such that  $|f(x) - r| \leq \varepsilon$  for all  $x$  in the unit square of  $F$ .  
(ii) Let  $B, f$  be as before. We denote by  $\gamma'(f)$  the collection of all  $r \in \mathbb{R}$  such that for any  $k$  and  $\varepsilon$  as above,  $F$  can be chosen to be  $(1 + \varepsilon)$ -isomorphic to a  $k$ -dimensional Hilbert space.

**Fact 3.4.** (*Dvoretzky-Milman Theorem*). Let  $f$  be a uniformly continuous function on the unit sphere of an infinite dimensional Banach space  $B$ . Then the spectrum  $\gamma(f)$  is not empty. Moreover,  $\gamma'(f)$  is not empty.

*Proof.* For the proof we refer the reader to e.g. [BL00], section 12 (specifically, combine Theorem 12.10 and Proposition 12.3 there). Alternatively, see [Pes06] for a detailed discussion of concentration phenomena. QED<sub>3.4</sub>

The first approximation to our goal is the following.

**Proposition 3.5.** Let  $\pi(x)$  be a wide partial type,  $A$  a set containing the domain of  $\pi$ ,  $\varphi(x, \bar{a})$  be a formula over  $A$ . Then there exists  $r \in \mathbb{R}$  such that the partial type  $\pi(x) \cup [\varphi(x, \bar{a}) = r]$  is wide.

*Proof.* Without loss of generality we may assume that  $\|x\|^\pi = 1$ .

Let  $B$  be an infinite dimensional subspace of  $\mathfrak{B}$  whose unit sphere  $\mathbf{S}(B)$  is contained in  $\pi^\mathfrak{C}$ . The formula  $\varphi(x, \bar{a})$  induces a uniformly continuous function  $f$  from  $\mathbf{S}(B)$  to  $\mathbb{R}$ . By Dvoretzky-Milman Theorem 3.4,  $\gamma'(f) \neq \emptyset$ . Let  $r \in \gamma'(f)$ .

Let  $H = \ell_2$ . For every  $v \in H$ , introduce a free variable  $x_v$ . Let  $\mathbf{x} = \langle x_v : v \in H \rangle$ . Denote by  $\Lambda(\mathbf{x})$  the linear quantifier free diagram of  $H$  with variables  $x_v$ . That is,

$$\Lambda(\mathbf{x}) = \{x_v = \sum_{i < k} \lambda_i x_{v_i} : v, v_i \in H, \lambda_i \in \mathbb{R}, v = \sum_{i < k} \lambda_i v_i\}$$

Consider the following collection of formulas. This is the (approximate) quantifier free diagram of  $H$  with the additional requirement that the unit sphere  $\mathbf{S}(\mathbf{x})$  satisfies  $\pi(x) \& [\varphi(x, \bar{a}) = r]$ .

$$\begin{aligned} \Gamma(\mathbf{x}) = & \Lambda(\mathbf{x}) \cup \{\pi(x_v) \& |\varphi(x_v, \bar{a}) - r| \leq \varepsilon : \|v\|_H = 1, \varepsilon > 0\} \cup \\ & \{(1 - \varepsilon)\|v\|_H \leq \|x_v\| \leq (1 + \varepsilon)\|v\|_H : v \in H, \varepsilon > 0\} \end{aligned}$$

We claim that  $\Gamma(\mathbf{x})$  is finitely satisfiable in  $\mathfrak{C}$ . Indeed, in order to make sure this is true, one has to argue that for any  $k$  and any  $\varepsilon > 0$  there is a  $k$ -dimensional subspace  $F$  of  $B$  which is  $(1 + \varepsilon)$ -isomorphic to the  $k$ -dimensional Hilbert space  $\ell_2^k$  such that  $\varphi(x, \bar{a}) \sim_\varepsilon r$  on  $\mathbf{S}(F)$ , and this follows immediately from the fact that  $r \in \gamma'(f)$ .

This shows that  $\pi(x) \cup \{\varphi(x, \bar{a}) = r\}$  is a wide type (in fact, it contains the unit ball of an infinite dimensional subspace isometric to  $\ell_2$ ). QED<sub>3.5</sub>

Modifying the proof of the Proposition above, we also obtain the following.

**Lemma 3.6.** *Let  $\langle \pi_i : i < \lambda \rangle$  be an increasing chain of wide partial types. Then  $\pi = \bigcup_{i < \lambda} \pi_i$  is wide.*

*Proof.* We use compactness as in the proof of Proposition 3.5. That is, let  $\Lambda(\mathbf{x})$  be as there, and let

$$\begin{aligned} \Gamma(\mathbf{x}) = & \Lambda(\mathbf{x}) \cup \{\pi_i(x_v) : \|v\|_H = 1, i < \lambda\} \cup \\ & \{(1 - \varepsilon)\|v\|_H \leq \|x_v\| \leq (1 + \varepsilon)\|v\|_H : v \in H, \varepsilon > 0\} \end{aligned}$$

Clearly  $\Gamma$  is finitely satisfiable, hence consistent, so the union  $\pi$  is wide. QED<sub>3.6</sub>

**Theorem 3.7.** *(Existence of Wide Types). Let  $\pi(x)$  be a wide partial type,  $A$  a set containing the domain of  $\pi$ ,  $\Delta$  a collection of formulae closed under connectives. Then there exists a complete  $\Delta$ -type  $p$  over  $A$  containing  $\pi$  which is wide.*

*Remark 3.8.* (i) We will normally use  $\Delta = L$  or  $L_{\mathfrak{B}}$  or  $\Delta =$  quantifier free formulae in  $L$  or  $L_{\mathfrak{B}}$ .

(ii) Note that since  $x = x$  is wide, the theorem implies in particular that there exists a complete wide type over any set.

*Proof.* Without loss of generality we may assume that  $\|x\|^\pi = 1$ .

Enumerate all  $\Delta$ -formulae over  $A$   $\langle \varphi_\alpha(x, \bar{a}_\alpha) : \alpha < \lambda = |A| + |T| \rangle$  such that

(\*) If  $\delta$  is a limit ordinal and  $\alpha_1 < \alpha_2 < \dots < \alpha_k < \delta$ , then for any  $k$ -ary connective  $F$ , for some  $\alpha < \delta$  we have

$$F(\varphi_{\alpha_1}(x, \bar{a}_{\alpha_1}), \dots, \varphi_{\alpha_k}(x, \bar{a}_{\alpha_k})) = \varphi_\alpha(x, \bar{a}_\alpha)$$

Now construct an increasing continuous sequence of wide types  $\pi_\alpha$  by induction on  $\alpha$  such that:

- $\pi_0(x) = \pi(x)$
- $\pi_\alpha(x)$  determines the value of  $\varphi_\beta(x, \bar{a}_\beta)$  for all  $\beta < \alpha$

For successor stages, use Proposition 3.5, and for limit stages apply compactness as in the proof of Lemma 3.6. This is possible by (\*) above.

Obviously  $p = \pi_\lambda$  is as required.

QED<sub>3.8</sub>

Analyzing the proof, we see that we have actually shown

**Corollary 3.9.** *Let  $\pi(x)$  be a wide partial type,  $A$  a set containing the domain of  $\pi$ ,  $\Delta$  a collection of formulae closed under connectives. Then there exists a complete  $\Delta$ -type  $p$  over  $A$  containing  $\pi$  such that  $\pi^\mathfrak{C}$  contains the unit sphere of an infinite dimensional subspace isometric to a Hilbert space.*

#### 4. WIDE STABLE TYPES

Let  $\Delta$  be a collection of formulas closed under connectives and permutations of variables.

In this section we will use the notion of the algebraic closure of a set  $A$ ,  $\text{acl}(A)$ , which is the collection of all  $b$  whose orbit under the action of the automorphism group  $\text{Aut}(\mathfrak{C}/A)$  is finite. Recall that any model is algebraically closed, that is,  $\text{acl}(M) = M$ .

Recall that a complete  $\Delta$ -type  $p$  is called *definable* if for every  $\Delta$ -formula  $\varphi(x, y)$ , the function  $\theta(y) = d_p x \varphi(x, y)$  defined as  $d_p x \varphi(x, a) = \varphi^p(x, a)$  is a definable predicate (that is, can be uniformly approximated by formulae). We say that  $\pi$  is  $\Delta$ -definable if for every  $\Delta$ -formula  $\varphi$ , the function  $d_p x \varphi(x, y)$  can be uniformly approximated by  $\Delta$ -formulae.

As an example, let  $\Delta$  be the collection of all quantifier free formulae in the language  $L_\mathfrak{B}$ , and let  $p$  be a complete quantifier free 1-type over a closed subspace  $A$ . So  $p$  is determined by conditions of the form  $\|x + a\| = r_a$  for all  $a \in A$ . In other words,  $p$  is determined by the function  $\tau_p: A \rightarrow \mathbb{R}$  defined by  $\tau(a) = \|x + a\|^p$ . We call  $p$  definable if this function is a definable predicate (that is, can be uniformly approximated by formulae), and we call it quantifier-free definable, if it can be uniformly approximated by quantifier-free formulae.

We will not use the notion of a stable formula (as defined in [BYU10]) in this article. Let us just remark that a formula  $\varphi(x, y)$  is stable if and only if every  $\varphi$ -type is  $\Delta$ -definable, where  $\Delta$  is the closure of  $\varphi$  under connectives and permutations of variables. For example, the norm  $\|x + y\|$  is stable in  $\mathfrak{C}$  (in the sense of Krivine and Maurey [KM81]) if and only if every quantifier-free type is quantifier-free definable. See [BYU10] (or [Iov99a] for a slightly less general formulation).

The following definition is a straightforward generalization of the classical concept due to Lascar and Poizat:

**Definition 4.1.** Let  $\Delta$  be a collection of formulae closed under connectives and permutations of variables. A partial  $\Delta$ -type  $\pi$  (possibly with parameters) is called  $\Delta$ -stable (or

simply *stable* when  $\Delta$  is clear from the context) if every extension of it to a  $\Delta$ -type over  $\mathfrak{C}$  is  $\Delta$ -definable.

*Remark 4.2.* In [Iov05] José Iovino studies quantifier free types over Banach spaces that he calls “stable”. We would like to alert the reader to the fact that Iovino’s concept is significantly weaker the classical notion defined above. In a dependent theory [She90, She04] (if one considers all formulae, and not just q.f. ones), Iovino’s definition is equivalent to a (much more general than stability) notion of *generic stability* [She04, Usv09, HP11]. In an arbitrary theory Iovino’s definition is even weaker than generic stability: e.g., a  $c_0$ -type is Iovino-stable, but not generically stable (for the discussion of generic stability in the general context see e.g. [PT11, GOU]).

We would like now to define *forking*. The following definition is equivalent to the classical one when one restrict their attention to stable types.

**Definition 4.3.** Let  $p \in S_\Delta(A)$  be a complete stable  $\Delta$ -type over  $A$ , and let  $\rho$  be a partial  $\Delta$ -type extending  $p$  (so  $\rho$  is stable as well). We say that  $\rho$  *does not fork over  $A$*  or *is a non-forking extension* of  $p$  if  $\rho$  is definable over  $\text{acl}(A)$ .

If  $\rho$  is not definable over  $\text{acl}(A)$ , we say that it forks over  $A$  (or is a forking extension of  $p$ ).

The following is a classical fact about stable types (a straightforward generalization of [LP79] to the continuous context).

**Fact 4.4.** *A complete stable type over an algebraically closed set is stationary, which means that it has a unique non-forking extension to a complete type over  $\mathfrak{C}$ .*

**Fact 4.5.** *Let  $p = p_0$  be a  $\Delta$ -stable type. Then there does not exist an increasing sequence of  $\Delta$ -types  $\langle p_i : i < |\Delta|^+ \rangle$  such that  $p_{i+1}$  is a forking extension of  $p_i$ .*

*Proof.* Denote  $\lambda = |\Delta|$ . Let  $q = \cup_{i < \lambda} p_i$ . Since  $q$  extends  $p = p_0$  and  $p$  is stable,  $q$  is  $\Delta$ -definable (hence definable over a subset  $B$  of  $\text{dom}(p)$  of cardinality  $\lambda$ ). Clearly  $B \subseteq \text{dom}(p_i)$  for some  $i$ ; but since  $p_{i+1} = q \upharpoonright \text{dom}(p_{i+1})$ , this implies that  $p_{i+1}$  is definable over  $B$ , hence is a nonforking extension of  $p_i$ , a contradiction. QED<sub>4.5</sub>

From now on, let us fix  $\Delta$  containing the quantifier free formulae of  $L_{\mathfrak{B}}$ , closed under connectives and permutations of variables. When we say “type”, “formula”, etc, we mean  $\Delta$ -type,  $\Delta$ -formula.

**Corollary 4.6.** *(Density of minimal types) Let  $\pi(x)$  be a partial wide type over a set  $A$ . Then there exists  $B \supseteq A$  with  $|B \setminus A| \leq |\Delta|$  and  $p \in S_\Delta(B)$  which is wide minimal, that is,  $p$  has a unique wide extension to  $\mathfrak{C}$ . Moreover, this unique extension is the unique nonforking extension of  $p$ . That is, no forking extension of  $p$  to a superset of  $B$  is wide.*

*Proof.* Construct by induction an increasing continuous sequence of sets  $A_i$  and an increasing sequence of types  $p \in S_\Delta(A_i)$  such that

- $A_0 = A$

- $|A_{i+1} \setminus A_i|$  is finite
- $p_0$  extends  $\pi$
- $p_i$  is wide for all  $i$
- $p_{i+1}$  forks over  $A_i$

Successor stages of the construction are clear. For limit stages, use Lemma 3.6. Since the construction has to get stuck at some  $i < |\Delta|^+$ , clearly (by stationarity)  $B = \text{acl}(A_i)$  and any extension of  $p_i$  to  $B$  are as required. QED<sub>4.6</sub>

We will now study the structure of minimal wide (stable) types.

Let  $O$  be a linearly ordered set. Recall that a sequence  $I = \langle a_i : i \in O \rangle$  is called  $\Delta$ -*indiscernible* over a set  $A$  if the  $\Delta$ -type of any finite sequence  $a_{i_1} \dots a_{i_k}$  over  $A$  depends only on the order between the indices  $i_1, \dots, i_k \in O$ . So if  $\Delta$  is the collection of all the quantifier free formulae in the language  $L_{\mathfrak{B}}$  and  $A = \emptyset$ , then  $I$  is  $\Delta$ -indiscernible if and only if  $I$  is 1-subsymmetric. As mentioned before, we will omit  $\Delta$ .

A sequence  $I$  as above is called an *indiscernible set* over  $A$  if the type of any finite sequence  $a_{i_1} \dots a_{i_k}$  over  $A$  depends only on the number  $k$ . As an example, one may think of the standard basis of  $\ell_p$ .

The following is another classical fact about stable types:

- Fact 4.7.** (i) *Let  $p$  be a stable type,  $I$  an indiscernible sequence of realizations of  $p$ . Then  $I$  is an indiscernible set.*
- (ii) *Let  $p$  be a stable type,  $A$  a set,  $I$  a sequence of realizations of  $p$  of length at least  $(|A| + |T|)^+$ . Then there exists an infinite subsequence  $I' \subseteq I$ , which is indiscernible over  $A$ .*

We now need to introduce the notion of a *Morley sequence*. In general, a *Morley sequence* in a type  $p \in S(A)$  is an indiscernible sequence  $I = \langle a_i : i < O \rangle$  of realizations of  $p$  such that  $\text{tp}(a_i / Aa_{<i})$  does not fork over  $A$ . Note that from stationarity of stable types, Fact 4.4, it follows that the only way to obtain a Morley sequence in a stable type  $p$  over an algebraically closed set  $A$  is as follows: let  $q$  be the unique global nonforking extension of  $p$ . Define  $\langle a_i : i < \omega \rangle$  such that  $a_i \models q \upharpoonright Aa_{<i}$ . One still needs to make sure that  $I$  is indiscernible over  $A$ , but this comes for free:

**Fact 4.8.** *Let  $q$  be a global type definable over a set  $A = \text{acl}(A)$ . Define a sequence  $I$  as described above. Then  $I$  is indiscernible over  $A$ .*

*Proof.* This is in fact true whenever  $q$  is invariant under the action of  $\text{Aut}(\mathfrak{C}/A)$ , see [She90], or [Usv] for an argument in continuous logic. QED<sub>4.8</sub>

**Definition 4.9.** Let  $\lambda$  be a cardinal. A *block* of  $\lambda$  is a finite subset of  $\lambda$ . For two blocks  $u_1, u_2$  of  $\lambda$  we say that  $u_1 < u_2$  if  $\max u_1 < \min u_2$ .

**Proposition 4.10.** (*Strong Uniqueness*) *Let  $p \in S_{\Delta}(A)$  be a minimal wide stable type, and let  $I = \langle a_{\alpha} : \alpha < \lambda \rangle$  be a Morley sequence in  $p$ . Then*

- (i)  *$I$  is an indiscernible set over  $A$ .*

- (ii) Let  $u_i$  be mutually disjoint blocks of  $\lambda$  for  $i < \omega$  and  $b_i \in \sum_{\alpha \in u_i} \mathbb{R}a_\alpha$  with  $\|b_i\| = 1$ . Then  $J = \langle b_i : i < \omega \rangle$  is an indiscernible set over  $A$  and a Morley sequence in  $p$ .

In particular,  $\text{tp}(J/A) = \text{tp}(I/A)$ .

*Proof.* (i) By stability (combine Fact 4.8 with Fact 4.7).

- (ii) Let  $p_\alpha = \text{tp}(a_\alpha/Aa_{<\alpha})$ . Fix  $\alpha < \lambda$ . Note that  $p_{\alpha+1}$  is a wide type extending  $p_\alpha$ . Let  $B$  be a subspace of infinite dimension, isometric to  $\ell_2$ , whose unit sphere is contained in  $p_\alpha^c$ . We may assume that  $a_\alpha \in \mathbf{S}(H)$ . Clearly for all  $a' \in \mathbf{S}(H)$  we have

$$\mathbb{R}a_\alpha + \mathbb{R}a' \subseteq B$$

Moreover, if  $r, r' \in \mathbb{R}$  are such that  $\|r + r'\|_2 = 1$ , then for all  $a' \in \mathbf{S}(H)$  we have

$$ra_\alpha + r'a' \in \mathbf{S}(H)$$

Hence the following partial type over  $Aa_{\leq\alpha}$  is wide:

$$\pi(x) = \{p(ra_\alpha + r'x) : r, r' \in \mathbb{R}, r^2 + (r')^2 = 1\}$$

By Theorem 3.7, there exists a wide complete type  $p'(x)$  over  $Aa_{\leq\alpha}$  extending  $\pi(x)$ . Since  $p'$  clearly extends  $p_\alpha$ , by minimality we get  $p' = p_{\alpha+1}$ , so  $a_\beta \models p$  for all  $\beta > \alpha$ . It follows by indiscernibility that for any  $\beta > \gamma \geq \alpha$  and  $r, r'$  with  $\|r + r'\|_2 = 1$ , we have  $ra_\gamma + r'a_\beta \models p_\alpha$ .

Moreover, by clause (i), that is, since  $I$  is an indiscernible set, it is easy to see that for any  $\beta > \gamma \geq \alpha$  and  $r, r'$  with  $\|r + r'\|_2 = 1$ , we have  $ra_\gamma + r'a_\beta \models \text{tp}(a_\alpha/Aa_{<\alpha}a_{>\beta}) = \text{tp}(a_\gamma/Aa_{<\alpha}a_{>\beta}) = \text{tp}(a_\beta/Aa_{<\alpha}a_{>\beta})$ . So denoting  $a' = ra_\gamma + r'a_\beta$ , we have that  $I' = a_{<\alpha} \frown a' \frown a_{>\beta}$  is a Morley sequence in  $p$ .

The case when  $a'$  is a general block element is proven by induction. That is, suppose that

$$a' = \sum_{i < n} r_i a_{\alpha_i} + r_n a_{\alpha_n}$$

such that  $\sum_{i \leq n} r_i^2 = 1$ . By the induction hypothesis, denoting

$$r'' = \left\| \sum_{i < n} r_i a_{\alpha_i} \right\| = \sqrt{\sum_{i < n} r_i^2}$$

and

$$a'' = \frac{1}{r''} \sum_{i < n} r_i a_{\alpha_i}$$

we have that the following sequence

$$I'' = a_{<\alpha_0} \frown a'' \frown a_{\alpha_n} \frown a_{>\alpha_n}$$

is a Morley sequence in  $p$ . Note that

$$(r'')^2 + r_n^2 = \sum_{i \leq n} r_i = 1$$

so by the case  $n = 2$  (which was our base case), the sequence

$$I' = a_{<\alpha_0} \frown (r''a'' + r_na_{\alpha_n}) \frown a_{>\alpha_n}$$

is a Morley sequence in  $p$ , as required.

Now it is easy to deduce the general statement of Strong Uniqueness by induction on the number of blocks.

QED<sub>4.10</sub>

**Proposition 4.11.** *Let  $p \in S_\Delta(A)$  be a minimal wide stable type, and let  $I = \langle a_\alpha : \alpha < \lambda \rangle$  be a Morley sequence in  $p$ . Then  $I$  is isometric to the standard basis of  $\ell_2$ . In other words, for every  $k < \omega$  and  $\lambda_0, \dots, \lambda_{k-1} \in \mathbb{R}$ , we have*

$$\left\| \sum_{i < k} \lambda_i a_i \right\|^2 = \sum_{i < k} |\lambda_i|^2$$

*Proof.* Let  $I'$  be (isometric to) the standard basis of an infinite dimensional  $\ell_2$  space,  $I' \subseteq p^\mathcal{C}$ . Since  $I'$  can be chosen as large as we want, by stability there is  $I \subseteq I'$  indiscernible over  $A$ . Clearly  $I$  is isometric to the standard basis of  $\ell_2$ . We need to show that  $I$  is a Morley sequence over  $A$ . Let  $H$  be the Hilbert space generated by  $I$ .

Without loss of generality  $A = \text{acl}(A)$ , so  $p$  is stationary. Let  $p^*$  be the global non-forking extension of  $p$ . Denote  $I = \langle a_i : i < \omega \rangle$ .

Let  $H_0$  be the subspace of  $H$  generated by  $Aa_0$ . Note that all elements of the unit sphere of  $(H_0)^\perp$  (the orthogonal complement in  $H$ ), which is an infinite-dimensional Hilbert space, satisfy the partial type

$$\pi(x) = p(x) \bigcup \{ \| \lambda_0 a_0 + \lambda x \|^2 = \lambda_0^2 + \lambda^2 : \lambda_0, \lambda \in \mathbb{R} \}$$

hence  $\pi(x)$  is wide. By Theorem 3.7, there exists  $q \in S(Aa_0)$  extending  $\pi(x)$ , which is wide. Since  $q$  extends  $p$  and is wide, by minimality of  $p$  we have  $q = p^* \upharpoonright Aa_0$ . Let  $b_0 = a_0, b_1 \models q$ . Then  $b_0, b_1$  start a Morley sequence in  $p$ , and as  $q$  extends  $\pi(x)$ , we see that  $\langle b_0, b_1 \rangle$  is isometric to the standard basis of a two-dimensional Hilbert space.

Now let  $\langle b_i : i < \omega \rangle$  be a Morley sequence in  $p$  continuing  $\langle b_i : i < 2 \rangle$ , and we show by induction on  $n$  that the sequence  $\langle b_i : i < n \rangle$  is isometric to the standard basis of an  $n$ -dimensional Hilbert space. Assuming that this holds for  $\langle b_i : i < n \rangle$ , let us take care of  $\langle b_i : i < n + 1 \rangle$ .

Let  $\langle \lambda_i : i < n + 1 \rangle$  be scalars in  $\mathbb{R}$ . By the induction hypothesis we have

$$(\diamond) \quad \left\| \sum_{i < n} \lambda_i b_i \right\|^2 = \sum_{i < n} \lambda_i^2$$



Denote

$$\lambda' = \sqrt{\sum_{i < n} \lambda_i^2} \quad \text{and} \quad b' = \frac{1}{\lambda'} \sum_{i < n} \lambda_i b_i$$

So  $\|b'\| = 1$ . By Strong Uniqueness (Proposition 4.10(ii)), the sequence  $\langle b', b_n \rangle$  is a (2-element) Morley sequence in  $p$ . By the induction hypothesis again (or by the case  $n = 2$ , which was our base case), we have

$$\left\| \sum_{i < n+1} \lambda_i b_i \right\|^2 = \|\lambda' b' + \lambda_n b_n\|^2 = (\lambda')^2 + \lambda_n^2 = \sum_{i < n} \lambda_i^2 + \lambda_n^2 = \sum_{i < n+1} \lambda_i^2$$

which completes the induction step.

QED<sub>4.11</sub>

## 5. ON HENSON'S CONJECTURE

We recall that throughout this paper we are assuming that  $K$  is an elementary class of Banach spaces with extra-structure,  $\mathfrak{C}$  its monster model. In this section we will also assume that the language of  $K$  (which we denote by  $L$ ) is *countable*.

Let  $M \in K$ ,  $A \subseteq M$ . We say that  $M$  is *prime* over  $A$  if whenever  $A \subseteq N \in K$ , there is an elementary embedding  $f: M \hookrightarrow N$  which is the identity on  $A$ .

We now state some standard facts about non-separably categorical continuous theories [SU11], [BY05].

**Fact 5.1.** *Assume that  $K$  is uncountably categorical. Let  $A \subseteq \mathfrak{C}$ . Then there exists a model  $M \in K$  which is prime over  $A$ .*

*Proof.* This is true in a more general context of  $\aleph_0$ -stable  $K$ . See section 4 of [SU11] or [BY05]. QED<sub>5.1</sub>

**Fact 5.2.** *Assume that  $K$  be uncountably categorical. Then  $K$  is  $\aleph_0$ -stable, in particular stable.*

Recall that  $K$  is stable if and only if every type in  $\mathfrak{C}$  in the language  $L$  is stable.

**Fact 5.3.** *(Morley's Theorem for continuous logic, [SU11, BY05]). Assume that  $K$  is uncountably categorical. Then  $K$  is categorical in every uncountable density. Moreover, every non-separable model in  $K$  is saturated.*

We are now ready to prove the main result of the paper.

**Theorem 5.4.** *Let  $K$  be an elementary class of Banach space with extra-structure, as defined in section 2, and assume that the language of  $K$  is countable. Equivalently, assume that  $T$  is a countable continuous theory whose monster model  $\mathfrak{C}$  expands a Banach space  $\mathfrak{B}$ .*

*Assume that  $K$  (equivalently,  $T$ ) is categorical in some uncountable density character. Then: There is a separable model  $M_0$  of  $T$  and a wide type  $p$  over  $M_0$  such that*

- Any Morley sequence in  $p$  is isometric to the standard orthonormal basis of a Hilbert space;
- Any non-separable model of  $T$  is prime over a Morley sequence in  $p$ .

Specifically, if  $M$  is a model of  $T$  of uncountable density character  $\lambda$ , then  $M$  is prime over a Morley sequence in  $p$  of length  $\lambda$ .

In particular, we have the following: Let  $B_0$  be the Banach space that underlines  $M_0$ . Let  $M \models T$  be of uncountable density character  $\lambda$ . Then there exists a spreading model  $H$  of  $M_0$  isometric to  $\ell_2(\lambda)$ , and  $M$  is prime over  $H$ .

*Proof.* By Fact 5.1, let  $\hat{M}_0$  be the prime model in  $K$  (prime over  $\emptyset$ ). By Theorem 3.7, there exists a wide type  $\hat{p}_0$  over  $\hat{M}_0$ . By Fact 5.2,  $K$  is stable, in particular the type  $\hat{p}_0$  is stable. By Corollary 4.6 (and e.g. Fact 5.1, although it is not needed for this), there is a separable model  $M_0 \in K$ ,  $\hat{M}_0 \prec M_0$ , and a minimal wide type extending  $\hat{p}_0$ .

Now let  $M \in K$  be of uncountable density  $\lambda$ . By Fact 5.3,  $M$  is  $\lambda$ -saturated, so we may assume that  $M_0 \subseteq M$ . By saturation again, there is a Morley sequence  $I = \langle a_i : i < \lambda \rangle$  in  $p_0$ ,  $I \subseteq M$ . Let  $M'$  be a prime model over  $I$  (Fact 5.1). Then  $M'$  has density  $\lambda$ ; since  $K$  is categorical in  $\lambda$  by Fact 5.3,  $M'$  is isometric to  $M$ .

So  $M$  is prime over a sequence isometric to a Morley sequence in  $p_0$ . The desired conclusion follows now from Proposition 4.11. QED<sub>5.4</sub>

*Remark 5.5.* In Theorem 5.4, one may assume that  $M_0$  is the saturated separable model of  $T$ ; however, not necessarily the prime model. It would be interesting to find out whether an  $\ell_2$  type exists over the prime model as well.

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